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First Passage Time Densities for Random Walk Spans

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A general expression is derived for the Laplace transform of the probability density of the first passage time for the span of a symmetric continuous-time random walk to reach level S. We show that when the mean time between steps is finite, the mean first passage time to S is proportional to S^2. When the pausing time density is asymptotic to a stable density we show that the first passage density is also asymptotically stable. Finally when the jump distribution of the random walk has the asymptotic form $p(j) \sim A/|j|^{\alpha+1}$, $0 < \alpha < 2$ it is shown that the mean first passage time to S goes like S^α.

KEY WORDS: Random walks, spans; stable laws.

1. INTRODUCTION

Let $x(t)$ be a stochastic process that is nondecreasing in time, and let $\tau(x)$ be the first passage time for x to reach level X. It is well known that the probability distributions of $x(t)$ and $\tau(x)$ are related by the identity

$$\Pr\{x(t) \leqslant X\} = \Pr\{\tau(X) \geqslant t\} \tag{1}$$

This duality has been used to derive limit theorems for renewal processes[1] and can be used to derive many of the known results relating to the maximum displacement of random walks on a line. In this note we apply a formalism based on Eq. (1) to discuss first passage times for the span of symmetric one-dimensional random walks. The span of such a random

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walk (sometimes referred to as the range) is defined in terms of the maximum and minimum displacement $x_{\max}(t)$ and $x_{\min}(t)$ as

$$s(t) = x_{\max}(t) - x_{\min}(t) \tag{2}$$

Clearly $s(t)$ is a nondecreasing stochastic process so that Eq. (1) can be applied to derive results related to the earliest times at which $s(t)$ reaches a target level that we denote by S. Properties of the span distribution were first derived by Daniels[2] and later by Feller[3] and Darling and Siegert.[4] Weiss and Rubin[5] extended this analysis to describe properties of the continuous time random walk (CTRW). Our current interest in this subject is based on the development of first passage time properties of reptating chains[6] as a model for the relaxation properties of polymer chains.

2. GENERAL FORMALISM

For simplicity we will analyze the span first passage times for random walks on a continuous line in discrete time and for CTRW's. One can easily develop analogous results for lattice random walks. Let $p(s; n)$ be the probability density for the span s, at step n. The probability that $s(n) \leqslant S$ is clearly

$$P(S; n) = \int_0^S p(s; n)\, ds \tag{3}$$

so that the probability that the first passage time is exactly equal to n is $P(S; n-1) - P(S; n)$. In order to calculate the analogous quantity for the CTRW we note that for such a walk the first passage time (for $S > 0$) must coincide with the time at which some step is made. Let $\psi(t)$ be the probability density for the time between two successive steps and let $\psi_n(t)$ be the probability density for the time at which the nth step is taken. If $\hat{\psi}(u)$ denotes the Laplace transform of $\psi(t)$ then the transform of $\psi_n(t)$ is $[\hat{\psi}(u)]^n$. One can formally write an expression for the probability density of interest as

$$h(S; t) = \sum_{n=1}^{\infty} [P(S; n-1) - P(s; n)]\, \psi_n(t), \qquad S > 0 \tag{4}$$

The form of this equation suggests that simplification will result if the Laplace transform is taken of both sides. If we take note of the fact that $P(S{:}0) = 1$ when $S > 0$ then one easily shows that

$$\hat{h}(S; u) = \hat{\psi}(u) - [1 - \hat{\psi}(u)] \sum_{n=1}^{\infty} P(S; n)\, \hat{\psi}^n(u) \tag{5}$$

where $P(S{:}n)$ is defined in Eq. (3).

3. APPROXIMATE RESULTS

The preceding result is exact, no approximations having been made. In order to make further progress we need to introduce an approximation for $p(S{:}n)$. Let us first consider random walks for which the variance of distance moved in a single step is $\sigma^2 < \infty$, i.e., is finite. For such random walks it has been shown[5] that the span density can be approximated by

$$p(s;n) \sim \frac{8\sigma^2 n}{s^3} \sum_{j=0}^{\infty} \left[\frac{\pi^2(2j+1)^2\sigma^2 n}{s^2} - 1 \right] \exp\left[-\frac{\pi^2\sigma^2(2j+1)^2 n}{2s^2} \right] \tag{6}$$

The integral in Eq. (3) can then be evaluated, leading to the result

$$P(S;n) \sim \frac{8}{\pi^2} \sum_{j=0}^{\infty} \left[\frac{1}{(2j+1)^2} + \frac{\pi^2\sigma^2 n}{S^2} \right] \exp\left[-\frac{\pi^2\sigma^2(2j+1)^2 n}{2S^2} \right] \tag{7}$$

When this is substituted into Eq. (5) one can readily justify an interchange of orders of integration over n and j, allowing us to evaluate the sum over n. The result of this calculation is

$$\hat{h}(S;u) \sim \hat{\psi}(u) - \frac{8}{\pi^2} [1 - \hat{\psi}(u)]\, \hat{\psi}(u) \sum_{j=0}^{\infty} \left\{ \frac{1}{(2j+1)^2} \frac{1}{\Gamma_j - \hat{\psi}(u)} + \frac{\pi^2\sigma^2}{S^2} \frac{\Gamma_j}{[\Gamma_j - \hat{\psi}(u)]^2} \right\} \tag{8}$$

in which

$$\Gamma_j = \exp\left[\frac{\pi^2\sigma^2(2j+1)^2}{2S^2} \right] \tag{9}$$

Let us now specialize to the case in which the target span, S, is much greater than the standard deviation of a single step of the random walk, σ, so that many steps are required for the span to reach S. In such a case we can write

$$\Gamma_j \sim 1 + \frac{\pi^2\sigma^2(2j+1)^2}{2S^2} \tag{10}$$

The substitution of this approximation into Eq. (8) allows us to evaluate the sum over j exactly[7] with the result that

$$\hat{h}(S;u) \sim \hat{\psi}(u)\, \mathrm{sech}^2 \left\{ \frac{S}{\sigma} \left[\frac{1 - \hat{\psi}(u)}{2} \right]^{1/2} \right\} \tag{11}$$

When $\psi(t)$ has a finite mean, T, this formula together with

$$\langle \tau(S) \rangle = -\left.\frac{\partial \hat{h}}{\partial u}\right|_{u=0+} \tag{12}$$

allows us to write

$$\langle \tau(S) \rangle = T\left(1+\frac{S^2}{2\sigma^2}\right) \sim \frac{TS^2}{2\sigma^2} \tag{13}$$

which is the result that one would expect since $\langle s(t) \rangle \sim t^{1/2}$ at large times. Let us next consider the case in which the average time between successive steps is infinite, and for large times

$$\psi(t) \sim T_0^\alpha / t^{\alpha+1}, \qquad 0<\alpha<1 \tag{14}$$

For this class of pausing time densities it is known that

$$\hat{\psi}(u) \sim 1-(uT_0)^\alpha \tag{15}$$

as $u \to 0$. If we substitute this into Eq. (11) we find that to lowest order

$$\begin{aligned} \hat{h}(S; u) &\sim 1-(uT_0)^\alpha\left(1+\frac{S^2}{2\sigma^2}\right) \\ &\sim \exp\left[-(uT_0)^\alpha\left(1+\frac{S^2}{2\sigma^2}\right)\right] \end{aligned} \tag{16}$$

This form for the Laplace transform allows us to identify the asymptotic form (in time) of $h(S; t)$ as a stable law. Specifically, if $f_\alpha(t)$ is the stable law density of order α whose Laplace transform can be written

$$\mathscr{L}\{f_\alpha(t)\} = \exp(-u^\alpha) \tag{17}$$

then the last line of Eq. (16) allows us make the identification

$$h(S; t) \sim f_\alpha(t/a)/a \tag{18}$$

where a is the constant

$$a = T_0\left(1+\frac{S^2}{2\sigma^2}\right)^{1/\alpha} \sim T_0\left(\frac{S^2}{2\sigma^2}\right)^{1/\alpha} \tag{19}$$

Thus $h(S; t)$ has the asymptotic form

$$h(S; t) \sim a^\alpha / t^{\alpha+1} \tag{20}$$

at sufficiently large values of t/T_0.

4. STABLE LAW JUMPS

In the cases considered so far we have assumed that the transition probabilities were such that the variance for a single step is finite. We conclude by considering the case in which the $p(j)$ themselves have a long tail of the form

$$p(j) \sim A/|j|^{\alpha+1}, \qquad 0<\alpha<2 \tag{21}$$

where A is a positive constant. For such random walks one knows that the structure function satisfies

$$\lambda(\theta) = \sum_{j=-\infty}^{\infty} p(j) \cos j\theta \sim 1 - |\theta/\theta_0|^{\alpha} \tag{22}$$

in the limit $\theta \to 0$. In this equation the constant θ_0 can be related to A in Eq. (21) by

$$\theta_0^{\alpha} = \frac{2\Gamma(1+\alpha)\sin(\pi\alpha/2)}{A} \tag{23}$$

The span density for such random walks has been found to be[5] given by

$$p(s; n) \sim \frac{8}{s^3} \sum_{j=0}^{\infty} \frac{d^2}{d\theta^2} \left[\exp\left(-\frac{n\theta^{\alpha}}{\theta_0^{\alpha}} \right) \right] \Bigg|_{\theta = (\pi(2j+1))/S} \tag{24}$$

for large $\theta_0 S$. If we denote the exponential appearing in this equation by $h(\theta)$ then the integral appearing in Eq. (3) is

$$\begin{aligned} \int_0^S p(s; n)\, ds &\sim 8 \sum_{j=0}^{\infty} \int_0^S \frac{1}{s^3} h'' \left[\frac{\pi(2j+1)}{s} \right] ds \\ &= \frac{8}{\pi^2} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \left\{ h\left[\frac{\pi(2j+1)}{S} \right] - \frac{\pi(2j+1)}{S} h' \left[\frac{\pi(2j+1)}{S} \right] \right\} \\ &= \frac{8}{\pi^2} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \left\{ 1 + n\alpha \left[\frac{\pi(2j+1)}{\theta_0 S} \right]^{\alpha} \right\} \exp\left\{ -n \left[\frac{\pi(2j+1)}{\theta_0 S} \right]^{\alpha} \right\} \end{aligned} \tag{25}$$

One can, as in our earlier analysis, interchange the orders of summation over j and n in Eq. (5) and evaluate the sum over n explicitly. Since Eq. (24) is only valid when $\theta_0 S \gg 1$ it follows that we can expand the exponential in the resulting expression keeping the lowest order term just as in Eq. (10). One finds, in this way, that $\hat{h}(S; u)$ can be expressed as the sum

$$\hat{h}(S; u) \sim \hat{\psi}(u) - \frac{8}{\pi^2}\hat{\psi}(u)[1 - \hat{\psi}(u)] \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \left\{ \frac{1}{1 - \hat{\psi}(u) + \Omega_j} + \frac{\alpha\Omega_j}{[1 + \hat{\psi}(u) + \Omega_j]^2} \right\} \tag{26}$$

where

$$\Omega_j = \left[\frac{\pi(2j+1)}{\theta_0 S}\right]^{\alpha} \tag{27}$$

On the assumption that the mean time between successive steps of the random walk is a finite constant T the mean first passage time to level S is, in this order of approximation,

$$\langle \tau(S) \rangle \sim \frac{8T}{\pi^2}(1+\alpha)\left(\frac{\theta_0 S}{\pi}\right)^{\alpha} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^{2+\alpha}} \tag{28}$$

Further extensions of the theory outlined here are possible. These include the development of an exact theory valid for lattice random walks as well as a theory that allows both for spatial transitions that have an infinite man value and a pausing time density with an infinite mean. These generalizations would not seem to have any obvious application, hence we do not give them here.

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